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Perturbative evaluation of the eigenvalues of the Herbst Hamiltonian

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Abstract

We reconsider the well-known and long-debated problem of the calculation of the eigenvalues of the Herbst Hamiltonian $2\sqrt{p^2+m^2}-\kappa/r$. We give a formulation of the problem that allows, for the first time, a perturbative evaluation of the eigenvalues for any n and l, and in principle up to any order in κ via standard Kato perturbation theory. We present the evaluation of the energy of the $n=1$ and $n=2$ states up to κ^6, confirming the result previously obtained by Le Yaouanc et al. with a completely different technique. Moreover we give the $n=2, l=1$ level, which is new. Discussion of the results and comparison with previous findings are given at the end.

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I. INTRODUCTION

In the last years the so-called Herbst Hamiltonian (sometimes also called the spinless relativistic Coulomb Hamiltonian):

$$H = 2\sqrt{\mathbf{p}^2 + m^2} - \frac{\kappa}{r}, \tag{1}$$

has been intensively studied in terms of both the spectrum and the eigenfunctions [1–6]. The reason of the interest in such a Hamiltonian is not only a mathematical one [1] but mainly a physical one, also due to the relevance of (1) in quarkonium phenomenology [7]. In fact we recall that, for an appropriate potential V the Hamiltonian $H = 2\sqrt{\mathbf{p}^2 + m^2} + V(\mathbf{x})$ is the spinless Salpeter equation representing a well-defined standard approximation to the Bethe–Salpeter formalism for the description of quark-antiquark bound states [5,8,9]; in this context Eq. (1) is suitable for the treatment of short-range effects in bound $q\bar{q}$ systems and for the study of decay rates; moreover it was also applied to the study of boson stars [10].

For the reasons presented above, a lot of attention has been addressed in the last years to the calculation of the ground state E_{10} of the Herbst Hamiltonian. First, an exact closed expression was found in Ref. [2] which, however, turned out to be wrong [4,2]; then, many analytic estimates of lower [1,10] and upper [5] bounds were given for E_{10}. Up to now we are aware of only a paper devoted to the perturbative calculation of E_{10} [6]. The method used in [6] has the merit of being systematic and allows the authors to settle an iteration procedure on a reasonably rigorous basis [1], but it is not general. Its starting point is a peculiar representation of the Hilbert space of $l = 0$ states (introduced in [2]) in which the Coulomb potential operator has a particularly simple form.

For this reason we have decided that it could be useful and instructive to apply a general

[1]The authors of Ref. [6] derive an equation for a function related to the eigenfunction and use it to obtain contributions up to κ^5 to the $l = 0$ levels. Then, they use the analyticity properties of the wave function to obtain next to leading orders up to κ^7.

and powerful perturbative technique to the calculation of the eigenvalues of the Herbst Hamiltonian. This method, not being founded on peculiar transformations, can in principle be applied for any value of the angular momentum and also extended to the consideration of potentials different from the Coulombian one. In fact it was proposed for the first time for the calculation of the positronium energy levels in Ref. [11].

The plan of the paper is as follows. In Sec. II we explain the perturbative technique; in Sec. III we apply it to the evaluation of the energy levels in the case $n = 1$ and $n = 2$ ($l = 0, 1$); in Sec. IV we make a comparison with the results previously obtained in the literature and draw some conclusions.

II. DESCRIPTION OF THE METHOD

We have to solve the equation

$$H\Phi = E\Phi \tag{2}$$

with the two-body relativistic Coulombian Hamiltonian given in (1) or equivalently with the one-body Hamiltonian

$$H = \sqrt{\mathbf{p}^2 + m^2} - \frac{\alpha}{r}. \tag{3}$$

The energy levels of the two-body (1) and the one-body (3) cases may easily be related by identifying both mass and Coulomb strength coupling parameters according to $m \to 2m$ and $\alpha \to \kappa/2$. In the following we consider the one-body case.

The perturbative solution of Eq. (2) which comes immediately to mind refers to the well-known non-relativistic Schrödinger Hamiltonian as the unperturbed starting point and supplies the eigenvalues E_{nl} ($n = 1, 2 \ldots; l = 0, 1, \ldots, n-1$) by evaluating perturbatively the expectation value of the relativistic corrections, coming from the expansion of the kinetic square root, on the analytically available zeroth-order eigenfunctions. This can be simply done for the contribution in the fourth power of $\mathbf{p}$. For instance, for the ground state zeroth-order energy E_{10}, we can calculate

$$\delta E_{10} = \langle\varphi_{100}| - \frac{\mathbf{p}^4}{8m^3}|\varphi_{100}\rangle = -\frac{5}{64}m\alpha^4. \tag{4}$$

where φ_{100} is the ground state Schrödinger–Coulomb wave-function:

$$\varphi_{100}(\mathbf{p}) = \frac{8\sqrt{\pi}(m\alpha)^{\frac{5}{2}}}{(\mathbf{p}^2 + (m\alpha)^2)^2}\,. \tag{5}$$

However, already with the sixth power of $\mathbf{p}$, one encounters a divergent integral; in order to get the next term in the perturbative expansion, which turns out to be of order α^5, one has to sum up an infinite number of contributions coming from all the possible intermediate states (cf. Ref. [6]). In general, the relativistic kinetic energy $\sqrt{\mathbf{p}^2 + m^2}$, behaving as $|\mathbf{p}|$ for very large $|\mathbf{p}|$, cannot be approximated by any finite polynomial in $\mathbf{p}^2$. This seems to prevent the possibility of profiting from the known results for the Schrödinger equation in the relativistic case. [2] There is, however, a trick for recovering, at least formally, a "kinetic energy" $\mathbf{p}^2$, even starting with $\sqrt{\mathbf{p}^2 + m^2}$; the price to pay is a kind of energy-dependent potential instead of the Coulomb one [11]. Let us consider Eq. (2) written in the momentum space

$$\int \frac{d^3\mathbf{q}}{(2\pi)^3}\left[(2\pi)^3\delta^3(\mathbf{p}-\mathbf{q})(\sqrt{\mathbf{p}^2 + m^2} - E) + V(\mathbf{p}-\mathbf{q})\right]\Phi(\mathbf{q}) = 0. \tag{6}$$

$$V(\mathbf{p}) = -\frac{4\pi\alpha}{\mathbf{p}^2}.$$

We define

$$R(E;\mathbf{p}) \equiv \left[\frac{2m}{E_p + E}\right]^{\frac{1}{2}}\;; \qquad E_p = \sqrt{\mathbf{p}^2 + m^2} \tag{7}$$

and put

$$\Phi(\mathbf{p}) = R^{-1}(E;\mathbf{p})\phi(\mathbf{p})$$

$$V(\mathbf{p}-\mathbf{q}) = R(E;\mathbf{p})v(E;\mathbf{p},\mathbf{q})R(E;\mathbf{q}). \tag{8}$$

By using Eqs. (7) and (8), we rewrite Eq. (6) in the form

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[2] At least, not without some ad hoc transformations and intricate calculations as in Ref. [6]

$$\int \frac{d^3\mathbf{q}}{(2\pi)^3}\left[(2\pi)^3\delta^3(\mathbf{p}-\mathbf{q})\left(\frac{\mathbf{p}^2}{2m}-\frac{E^2-m^2}{2m}\right)+v(E;\mathbf{p},\mathbf{q})\right]\phi(\mathbf{q})=0 \tag{9}$$

which is formally of the Schrödinger type, i.e. with a quadratic kinetic term, *but* with an energy-dependent interaction. Equation (9) is the appropriate starting point for the application of a perturbative technique.

Let us rephrase the same transformation in the propagator formalism, which is more appropriate to treat an energy-dependent perturbation term. With reference to Eq. (6) we can write for the one-particle propagator G :

$$G(E;\mathbf{p},\mathbf{q})=G_0(E;\mathbf{p})\left[(2\pi)^3\delta^3(\mathbf{p}-\mathbf{q})+\int\frac{d^3\mathbf{k}}{(2\pi)^3}V(\mathbf{p}-\mathbf{k})G(E;\mathbf{k},\mathbf{q})\right], \tag{10}$$

G_0 being the free propagator

$$G_0(E;\mathbf{p})=\frac{1}{E-E_p}. \tag{11}$$

Now, by using the identity

$$\frac{1}{E-E_p}=\frac{1}{(E^2-m^2)/2m-\mathbf{p}^2/2m}\left(\frac{E+E_p}{2m}\right), \tag{12}$$

Eqs. (8), and defining

$$g_0(E^*)=\frac{1}{E^*-\mathbf{p}^2/2m}\,,\qquad E^*=\frac{E^2-m^2}{2m}, \tag{13}$$

we can rewrite Eq. (10) as

$$g(E^*;\mathbf{p},\mathbf{q})=g_0(E^*;\mathbf{p})\left[(2\pi)^3\delta^3(\mathbf{p}-\mathbf{q})+\int\frac{d^3\mathbf{k}}{(2\pi)^3}v(E;\mathbf{p},\mathbf{q})g(E^*;\mathbf{k},\mathbf{q}),\right] \tag{14}$$

where v is given in (8) and

$$g(E^*;\mathbf{p},\mathbf{q})=R(E;\mathbf{p})G(E;\mathbf{p},\mathbf{q})R(E;\mathbf{q}). \tag{15}$$

We emphasize that in Eq. (14) the propagators g and g_0 has to be understood as depending on the modified energy E^*, which is a function of E.

Inserting V in place of v in (14) we obtain the exactly solvable equation for the well-known Schrödinger-Coulomb propagator g_s :

$$g_s(E^*;\mathbf{p},\mathbf{q}) = g_0(E^*;\mathbf{p})\left[(2\pi)^3\delta^3(\mathbf{p}-\mathbf{q}) + \int \frac{d^3\mathbf{k}}{(2\pi)^3} V(\mathbf{p}-\mathbf{k}) g_s(E^*;\mathbf{k},\mathbf{q})\right]. \tag{16}$$

The g_s propagator has poles in correspondence of the energy values

$$E^* = -\frac{m\alpha^2}{2n^2} \quad \Rightarrow E = E_n^0 \equiv \sqrt{m^2-\gamma_n^2}, \qquad \gamma_n \equiv \frac{m\alpha}{n}, \quad n = 1,2,\dots \quad , \tag{17}$$

and can be written for any n as

$$\begin{aligned} g_s(E^*;\mathbf{p},\mathbf{q}) &= \frac{\sum_{l,m}\varphi_{nlm}(\mathbf{p})\varphi^*_{nlm}(\mathbf{q})}{E^* + m\alpha^2/2n^2} + \hat{g}_n(E^*;\mathbf{p},\mathbf{q}), \\ &= \frac{m}{E_n^0}\frac{\sum_{l,m}\varphi_{nlm}(\mathbf{p})\varphi^*_{nlm}(\mathbf{q})}{E-E_n^0} - \frac{m}{E_n^0}\frac{\sum_{l,m}\varphi_{nlm}(\mathbf{p})\varphi^*_{nlm}(\mathbf{q})}{E+E_n^0} + \hat{g}_n(E^*;\mathbf{p},\mathbf{q}) \\ &\equiv \frac{m}{E_n^0}\frac{\sum_{l,m}\varphi_{nlm}(\mathbf{p})\varphi^*_{nlm}(\mathbf{q})}{E-E_n^0} + \hat{g}'_n(E;\mathbf{p},\mathbf{q}), \end{aligned} \tag{18}$$

φ_{nlm} being the Schrödinger–Coulomb wave-functions and $\hat{g}_n$ (or $\hat{g}'_n$) being the regular part of the propagator.

Now, by using standard Kato perturbation theory [12], one obtains the following expansion for the energy levels ($\delta V \equiv v - V$),

$$\begin{aligned} E_{nl} = E_n^0 &+ \frac{1}{2l+1}\langle \delta V(E_n^0)\rangle_{nl} + \\ &+ \frac{1}{2l+1}\left\{\langle \delta V(E_n^0)\hat{g}'_n(E_n^0)\delta V(E_n^0)\rangle_{nl} + \frac{1}{2l+1}\langle \delta V(E_n^0)\rangle_{nl}\left\langle \frac{\partial}{\partial E}\delta V(E_n^0)\right\rangle_{nl}\right\} + \\ &+ \frac{1}{2l+1}\langle \delta V(E_n^0)\hat{g}'_n(E_n^0)\delta V(E_n^0)\hat{g}'_n(E_n^0)\delta V(E_n^0)\rangle_{nl} + \dots \quad , \end{aligned} \tag{19}$$

where the symbol $\langle\ \rangle_{nl}$ stands for

$$\langle\ \rangle_{nl} \equiv \frac{m}{E_n^0}\sum_m \int \frac{d^3\mathbf{p}}{(2\pi)^3}\int \frac{d^3\mathbf{q}}{(2\pi)^3}\big(\quad\big)\varphi_{nlm}(\mathbf{p})\varphi^*_{nlm}(\mathbf{q}). \tag{20}$$

The summation over the quantic number m and the factors $1/(2l+1)$ takes in account the degeneration of the level.

Expansion (19) is valid for any n and any l. As higher order in δV supply higher-order leading contributions in α, Eq. (19) is perturbative not only in δV but also in α. So up to a given order in α only a finite number of terms in (19) contribute to it.

In the next section we apply Eq. (19) to the evaluation of E_{nl} in the cases $n = 1$ and $n = 2$ $l = 0, 1$ up to the order α^6.

III. CALCULATION OF THE ENERGY LEVELS

Up to order α^6, only the terms explicitly written in Eq. (19) have to be taken in to account. It is useful to write $\hat{g}'(E_n^0)$ in the form:

$$\hat{g}'(E_n^0) = \hat{g}\left(\frac{-\gamma_n^2}{2m}\right) - \frac{m}{2E_n^{0\,2}}\sum_{l,m}\varphi_{nlm}\varphi^*_{nlm} = \\ = g_0\left(\frac{-\gamma_n^2}{2m}\right) + g_0\left(\frac{-\gamma_n^2}{2m}\right) V\, g_0\left(\frac{-\gamma_n^2}{2m}\right) + \hat{R}'_n(E_n^0), \tag{21}$$

where

$$\hat{R}'_n(E_n^0) \equiv \hat{R}_n\left(\frac{-\gamma_n^2}{2m}\right) - \frac{m}{2E_n^{0\,2}}\sum_{l,m}\varphi_{nlm}\varphi^*_{nlm}.$$

The functions $R_1\left(-\gamma_1^2/2m\right)$ and $R_2\left(-\gamma_2^2/2m\right)$ are given in the appendix. From Eq. (21) and

$$\langle\delta V(E_n^0)\hat{g}'_n(E_n^0)\delta V(E_n^0)\hat{g}'_n(E_n^0)\delta V(E_n^0)\rangle_{nl} = \\ = \left\langle \delta V(E_n^0)\, g_0\left(\frac{-\gamma_n^2}{2m}\right)\, \delta V(E_n^0)\, g_0\left(\frac{-\gamma_n^2}{2m}\right)\, \delta V(E_n^0)\right\rangle_{nl} + o(\alpha^6),$$

we obtain the only relevant contributions to the energy levels up to order α^6 :

$$E_{nl} = E_n^0 + \frac{1}{2l+1}\langle\delta V(E_n^0)\rangle_{nl} + \frac{1}{2l+1}\Bigg\{\left\langle \delta V(E_n^0)\, g_0\left(\frac{-\gamma_n^2}{2m}\right)\, \delta V(E_n^0)\right\rangle_{nl} + \\ + \left\langle \delta V(E_n^0)\, g_0\left(\frac{-\gamma_n^2}{2m}\right) V\, g_0\left(\frac{-\gamma_n^2}{2m}\right)\, \delta V(E_n^0)\right\rangle_{nl} + \langle\delta V(E_n^0)\hat{R}'_n(E_n^0)\delta V(E_n^0)\rangle_{nl} + \\ + \frac{1}{2l+1}\langle\delta V(E_n^0)\rangle_{nl}\left\langle\frac{\partial}{\partial E}\delta V(E_n^0)\right\rangle_{nl}\Bigg\} + \\ + \frac{1}{2l+1}\left\langle \delta V(E_n^0)\, g_0\left(\frac{-\gamma_n^2}{2m}\right)\, \delta V(E_n^0)\, g_0\left(\frac{-\gamma_n^2}{2m}\right)\, \delta V(E_n^0)\right\rangle_{nl} + o(\alpha^6). \tag{22}$$

With the symbol $o(\alpha^6)$ we indicate higher-order contributions, typically starting with α^7 and $\alpha^7\ln\alpha$ terms.

First let us consider the case $l = 0$. Using the Schrödinger equation ($g_0 V\varphi = \varphi$) we can perform some straightforward cancellations and reduce Eq. (22) to the form:

$$E_{n0} = E_n^0 + \left\langle v(E_n^0)\, g_0\left(\frac{-\gamma_n^2}{2m}\right)\, v(E_n^0)\, g_0\left(\frac{-\gamma_n^2}{2m}\right)\, \delta V(E_n^0)\right\rangle_{n0} + \left\langle\frac{\partial}{\partial E}\delta V(E_n^0)\right\rangle_{n0}\langle\delta V(E_n^0)\rangle_{n0} \\ + \langle\delta V(E_n^0)\hat{R}'_n(E_n^0)\delta V(E_n^0)\rangle_{n0} + o(\alpha^6). \tag{23}$$

For the level $n = 1$ we obtain:

$$\begin{aligned}
\frac{\langle v g_0 v g_0 \delta V\rangle_{10}}{m} &= -\frac{1}{2}\alpha^4 + \frac{8}{3}\frac{\alpha^5}{\pi} - \alpha^6 \ln \alpha^{-1} + \\
&+ \left(\frac{1}{8} - \frac{1}{4}\zeta(3) - \frac{2}{\pi^2} + \frac{7}{\pi^2}\zeta(3) - \frac{\pi^2}{8}\right)\alpha^6 + o(\alpha^6), \\
\frac{\langle \delta V\rangle_{10}\langle \frac{\partial}{\partial E}\delta V\rangle_{10}}{m} &= \frac{1}{4}\alpha^6 + o(\alpha^6), \\
\frac{\langle \delta V \hat{R}'_1 \delta V\rangle_{10}}{m} &= \left(\frac{\pi^2}{8} - \frac{19}{8} + \frac{1}{4}\zeta(3)\right)\alpha^6 + o(\alpha^6).
\end{aligned}$$

Inserting the various contributions in Eq. (23), we obtain the final result:

$$\frac{E_{10}}{m} = 1 - \frac{1}{2}\alpha^2 - \frac{5}{8}\alpha^4 + \frac{8}{3}\frac{\alpha^5}{\pi} - \alpha^6 \ln \alpha^{-1} + \left(\frac{7}{\pi^2}\zeta(3) - \frac{2}{\pi^2} - \frac{33}{16}\right)\alpha^6 + o(\alpha^6). \tag{24}$$

Similarly for the $n = 2, l = 0$ case we calculate

$$\begin{aligned}
\frac{\langle v g_0 v g_0 \delta V\rangle_{20}}{m} &= -\frac{3}{32}\alpha^4 + \frac{1}{3}\frac{\alpha^5}{\pi} - \frac{1}{8}\alpha^6 \ln \alpha^{-1} + \\
&+ \left(\frac{127}{512} - \frac{1}{8}\zeta(3) - \frac{1}{4}\frac{1}{\pi^2} + \frac{7}{8}\frac{1}{\pi^2}\zeta(3) - \frac{\pi^2}{32} - \frac{1}{8}\ln 2\right)\alpha^6 + o(\alpha^6), \\
\frac{\langle \delta V\rangle_{20}\langle \frac{\partial}{\partial E}\delta V\rangle_{20}}{m} &= \frac{3}{256}\alpha^6 + o(\alpha^6), \\
\frac{\langle \delta V \hat{R}'_2 \delta V\rangle_{20}}{m} &= \left(\frac{1}{32}\pi^2 - \frac{231}{512} + \frac{1}{8}\zeta(3)\right)\alpha^6 + o(\alpha^6),
\end{aligned}$$

and then obtain

$$\begin{aligned}
\frac{E_{20}}{m} = 1 - \frac{1}{8}\alpha^2 - \frac{13}{128}\alpha^4 + \frac{1}{3}\frac{\alpha^5}{\pi} - \frac{1}{8}\alpha^6 \ln \alpha^{-1} + \\
+ \left(\frac{7}{8}\frac{1}{\pi^2}\zeta(3) - \frac{1}{4}\frac{1}{\pi^2} - \frac{197}{1024} - \frac{1}{8}\ln 2\right)\alpha^6 + o(\alpha^6).
\end{aligned} \tag{25}$$

When l is different from zero, we have to apply directly Eq. (22). Let us consider the case $n = 2, l = 1$. Then

$$\begin{aligned}
\frac{\langle \delta V\rangle_{21}}{m} &= -\frac{1}{32}\alpha^4 + \frac{19}{512}\alpha^6 + o(\alpha^6), \\
\frac{\langle \delta V g_0 \delta V\rangle_{21}}{m} &= \left(\frac{1}{96}\pi^2 - \frac{15}{128}\right)\alpha^6 + o(\alpha^6), \\
\frac{\langle \delta V g_0 V g_0 \delta V\rangle_{21}}{m} &= \left(\frac{1}{48}\pi^2 + \frac{1}{8}\zeta(3) - \frac{93}{256}\right)\alpha^6 + o(\alpha^6), \\
\frac{\langle \delta V \hat{R}'_2 \delta V\rangle_{21}}{m} &= \left(\frac{2051}{4608} - \frac{1}{8}\zeta(3) - \frac{1}{32}\pi^2\right)\alpha^6 + o(\alpha^6).
\end{aligned}$$

$$\frac{\langle \delta V \rangle_{21} \langle \frac{\partial}{\partial E} \delta V \rangle_{21}}{m} = \frac{3}{256}\alpha^6 + o(\alpha^6),$$
$$\frac{\langle \delta V g_0 \delta V g_0 \delta V \rangle_{21}}{m} = o(\alpha^6).$$

and, summing up all the contributions, we obtain

$$\frac{E_{21}}{m} = 1 - \frac{1}{8}\alpha^2 - \frac{7}{384}\alpha^4 + \frac{25}{27648}\alpha^6 + o(\alpha^6), \tag{26}$$

which, surprisingly, does not contain contributions in α^5 and $\alpha^6 \ln \alpha$.

The description (in another context) of some techniques used to extract from the integrals the various order in powers of α can be found in [13]. All calculations were performed with the help of the program of symbolic manipulations FORM [14].

IV. DISCUSSION AND CONCLUSIONS

In conclusion, in the case $n = 1, 2$, $l = 0$, we have obtained a result (Eqs. (24) and (25)) that confirms, up to order α^6, the previous findings of Le Yaouanc et al. [6]. However, the result obtained in the case $n = 2, l = 1$ (Eq. (26)) is new and displays also a different analytical structure.

We emphasize again that our method is general and can be applied to the calculation of the eigenvalue of the Herbst Hamiltonian for any n and any l, and in principle up to the desired order in α. Moreover, due to its generality, this method could be applied also in the case of a potential different from the Coulomb one. We notice that the problem can be solved on a general ground and by well-known energy-dependent perturbation theory due to the appropriate formulation of the starting point in (9). In this way, in fact, we gain an energy-dependent perturbation containing α.

Finally let us mention some other points:

1) The result we have obtained for the ground-state energy fits nicely in the upper and lower variational bounds given in Refs. [5,10].

2) The perturbative expansion of the eigenvalues given in (24)–(26) is suitable to obtain good numerical estimates of the energy of the level in principle up to the critical value of the coupling constant $\alpha = 2/\pi$. As an example, we present in Table 1 the evaluation of E_{10}, E_{20} and E_{21} for different values of α. We have chosen the same values of α as in Ref. [10], so that it could be appreciated (for small couplings at least) the improvement introduced by this perturbative expansion with respect to the variational evaluation (cf. the variational upper $E_{10}^{\max}$ and lower $E_{10}^{\min}$ estimates given in Table 2 of Ref. [10] and transcribed here in columns 2 and 3 of Table 1). In Table 1, the last figure in our results has been put in parenthesis because it is affected by the error coming from the neglected α^7 and $\alpha^7 \ln \alpha$ contributions. For $\alpha > 0.5$ the error obviously grows and we have therefore not considered significant to present the results and to make a comparison also in this case.

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APPENDIX A:

The non-singular part $\hat{g}_n$ of the Schrödinger–Coulomb propagator is obtained by expanding the analytical expression of g_s in the vicinity of the energy level and by subtracting the pole contribution. The expression of $\hat{g}_1$ evaluated in $-\gamma_1^2/2m$ reads (cf. [15]):

$$\hat{g}_1\left(\frac{-\gamma_1^2}{2m};\mathbf{p},\mathbf{q}\right) = \frac{(2\pi)^3\delta^3(\mathbf{p}-\mathbf{q})}{-\gamma_1^2/2m-\mathbf{p}^2/2m} + \frac{1}{-\gamma_1^2/2m-\mathbf{p}^2/2m}V(\mathbf{p}-\mathbf{q})\frac{1}{-\gamma_1^2/2m-\mathbf{q}^2/2m} - \\ - \frac{64\pi m\gamma_1^3}{(\mathbf{p}^2+\gamma_1^2)^2(\mathbf{q}^2+\gamma_1^2)^2}\left(\frac{5}{2} - 4\frac{\gamma_1^2}{\mathbf{p}^2+\gamma_1^2} - 4\frac{\gamma_1^2}{\mathbf{q}^2+\gamma_1^2}\right) -$$

$$
\begin{aligned}
&- \frac{64\pi m\gamma_1^3}{(\mathbf{p}^2+\gamma_1^2)^2(\mathbf{q}^2+\gamma_1^2)^2}\left\{\frac{1}{2}\ln C_1 + \frac{2C_1-1}{\sqrt{4C_1-1}}\mathrm{arctg}\sqrt{4C_1-1}\right\} \\
&\equiv g_0\left(-\frac{\gamma_1^2}{2m}\right) + g_0\left(-\frac{\gamma_1^2}{2m}\right) V\, g_0\left(-\frac{\gamma_1^2}{2m}\right) + \hat{R}_1\left(-\frac{\gamma_1^2}{2m}\right). \qquad \text{(A1)}
\end{aligned}
$$

In a similar way one obtains the regular part corresponding to the $n = 2$ pole, which evaluated in $-\gamma_2^2/2m$ reads:

$$
\begin{aligned}
\hat{g}_2\left(\frac{-\gamma_2^2}{2m};\mathbf{p},\mathbf{q}\right) &= \frac{(2\pi)^3\delta^3(\mathbf{p}-\mathbf{q})}{-\gamma_2^2/2m-\mathbf{p}^2/2m} + \frac{1}{-\gamma_2^2/2m-\mathbf{p}^2/2m}V(\mathbf{p}-\mathbf{q})\frac{1}{-\gamma_2^2/2m-\mathbf{q}^2/2m} - \\
&- \frac{256\pi m\gamma_2^3}{(\mathbf{p}^2+\gamma_2^2)^2(\mathbf{q}^2+\gamma_2^2)^2}\Bigg\{\frac{2\gamma_2^2(\mathbf{p}-\mathbf{q})^2}{(\mathbf{p}^2+\gamma_2^2)(\mathbf{q}^2+\gamma_2^2)}\left(-\frac{9}{2}+\frac{6\gamma_2^2}{\mathbf{p}^2+\gamma_2^2}+\frac{6\gamma_2^2}{\mathbf{q}^2+\gamma_2^2}\right)+ \\
&\qquad +\frac{3}{2}-\frac{4\gamma_2^2}{\mathbf{p}^2+\gamma_2^2}-\frac{4\gamma_2^2}{\mathbf{q}^2+\gamma_2^2}- \\
&\qquad -\left(\frac{1}{2C_2}-1\right)\ln C_2 + \frac{2C_2-4+1/C_2}{\sqrt{4C_2-1}}\,\mathrm{arctg}\sqrt{4C_2-1}\Bigg\} \\
&\equiv g_0\left(-\frac{\gamma_2^2}{2m}\right) + g_0\left(-\frac{\gamma_2^2}{2m}\right) V\, g_0\left(-\frac{\gamma_2^2}{2m}\right) + \hat{R}_2\left(-\frac{\gamma_2^2}{2m}\right), \qquad \text{(A2)}
\end{aligned}
$$

with

$$
C_n = \frac{(\mathbf{p}^2+\gamma_n^2)(\mathbf{q}^2+\gamma_n^2)}{4\gamma_n^2(\mathbf{p}-\mathbf{q})^2},
$$

and γ_n given in Eq. (17).

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TABLES

TABLE I. Calculation of the energy levels given in Eqs. (24),(25),(26). In columns 2 and 3 are shown the minimum $E_{10}^{\rm min}$ and the maximum $E_{10}^{\rm max}$ variational estimates of the ground-state energy as given in Table 2 of [10] are shown. The figure in parenthesis is of the same order as the uncalculated α^7 and $\alpha^7 \ln \alpha$ contributions.

α	$E_{10}^{\rm min}/m$[10]	$E_{10}^{\rm max}/m$[10]	E_{10}/m	E_{20}/m	E_{21}/m
0.0155522	0.9998785	0.9998791	0.99987902866(7)	0.99996976027(9)	0.99996976506(8)
0.1425460	0.989458	0.989613	0.98960(4)	0.99742(0)	0.99745(2)
0.2599358	0.96309	0.96364	0.9635(1)	0.9911(0)	0.9914(7)
0.3566678	0.92578	0.92673	0.926(1)	0.982(3)	0.983(8)
0.4359255	0.88013	0.88139	0.88(0)	0.97(2)	0.97(5)
0.5000000	0.82758	0.82910	0.82(9)	0.96(1)	0.96(7)